

Description of spectra of quadratic Pisot units

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Abstract

The spectrum of a real number $\beta > 1$ is the set $X^m(\beta)$ of $p(\beta)$ where p ranges over all polynomials with coefficients restricted to $\mathcal{A} = \{0, 1, \dots, m\}$. For a quadratic Pisot unit β , we determine the values of all distances between consecutive points and their corresponding frequencies, by recasting the spectra in the frame of the cut-and-project scheme. We also show that shifting the set \mathcal{A} of digits so that it contains at least one negative element, or considering negative base $-\beta$ instead of β , the gap sequence of the generalized spectrum is a coding of an exchange of three intervals.

Keywords: Pisot numbers, spectrum, interval exchange

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1. Introduction

The spectrum of a real number $\beta > 1$ is the set of $p(\beta)$ where p ranges over all polynomials with coefficients restricted to a finite set of consecutive integers, in particular,

$$X^m(\beta) = \left\{ \sum_{j=0}^n a_j \beta^j : n \in \mathbb{N}, a_j \in \{0, 1, \dots, m\} \right\}. \quad (1)$$

The study of such sets for $\beta \in (1, 2)$ and $m = 1$ was initiated by Erdős et al. in 1990 [6]. Their interest [7, 8] is to study the difference sequence $(y_{k+1} - y_k)_{k \in \mathbb{N}}$, where $X^m(\beta) = \{0 = y_0 < y_1 < y_2 < \dots\}$, in particular the values

$$l^m(\beta) = \liminf_{k \rightarrow \infty} (y_{k+1} - y_k) \quad \text{and} \quad L^m(\beta) = \limsup_{k \rightarrow \infty} (y_{k+1} - y_k).$$

The relevance of Pisot numbers in the problem of spectra was indicated by Bugeaud in 1996 [3] who showed that a real $\beta \in (1, 2)$ is a Pisot number if and only if $l^m(\beta) > 0$ for all $m \geq 1$. An important recent contribution is due to Feng [9] who shows that $l^m(\beta)$ is positive if and only if β is a Pisot number

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or $m < \beta - 1$. Many other authors have contributed to the problem; for an exhaustive overview, see the paper by Akiyama and Komornik [1].

A particular question is to determine the exact values of $l^m(\beta)$, $L^m(\beta)$. First to give such a result for general $m \geq 1$ were Komornik et al. in 2000 [17] who provided $l^m(\beta)$ for the golden ratio $\beta = \frac{1}{2}(1 + \sqrt{5})$. Komatsu in 2002 [16] and independently Borwein and Hare in 2003 [2] extended the result to all quadratic Pisot units. Their method, however, does not testify about the structure of the gap sequence $(y_{k+1} - y_k)_{k \in \mathbb{N}}$, in particular, nor about $L^m(\beta)$.

In 2002, Bugeaud [4] provides a substitution that can be used for generating the difference sequence for the spectrum $X^1(\beta)$ of the so-called d -bonacci numbers $\beta > 1$, zeros of $x^d - x^{d-1} - \dots - x - 1$. This allows him to determine all gaps and the corresponding frequencies. In the same year, Feng and Wen [10] showed that for any Pisot number β and $m > \beta - 1$, the sequence of distances $(y_{k+1} - y_k)_{k \in \mathbb{N}}$ in $X^m(\beta)$ can be generated by a substitution over a finite alphabet. Their proof is constructive, however, does not provide any explicit prescription for the substitution nor for the values of distances and their frequencies. Moreover, the cardinality of the alphabet of the substitution found by their construction grows rapidly with m . The construction of Feng and Wen was used in 2006 by Garth and Hare [12] for determining the substitution, gap sizes and their frequencies explicitly for the spectra $X^{\lfloor \beta \rfloor}(\beta)$, where $\beta > 1$ is a zero of $x^d - px^{d-1} - \dots - px - q$, $p \geq q \geq 1$.

Our main result concerns the distance sequence for the spectra $X^m(\beta)$ of quadratic Pisot units β . Unlike the previous results, we consider arbitrary $m \in \mathbb{N}$, $m > \beta - 1$. We show that the distances $y_{k+1} - y_k$ take (up to finitely many exceptions) only three values.

Theorem 1. *Let β be a quadratic Pisot unit, $m \in \mathbb{N}$, $m > \beta - 1$. Then there exist $\Delta_1, \Delta_2 > 0$ such that the distances $y_{k+1} - y_k$ between consecutive points in $X^m(\beta)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$, up to finitely many exceptions.*

The explicit values of Δ_1, Δ_2 and the frequencies of the gaps $\Delta_1, \Delta_2, \Delta_1 + \Delta_2$, dependently on m , are given in Theorem 21. Let us mention that for the particular case of the golden ratio, and any $m \geq 1$, this was already given in [14]. However, the proof given by the author contains a flaw, as explained in Section 3 of the present paper.

Our method in proof of Theorem 1 is the use of infinite words coding exchange of three intervals, the so-called 3iet words. More precisely, we show that the sequence of gaps in $X^m(\beta)$ ‘almost’ coincides with some 3iet word. Although the coincidence is broken at infinitely many places (see Proposition 7), the frequency of such perturbations is 0. This is the reason why the values for gaps and their frequencies given in [14] are correct.

It is interesting to mention that a perfect coincidence with 3iet words can be achieved when considering a slightly generalized problem, in particular, when shifting the alphabet to contain both positive and negative digits, or considering negative base. With such assumptions, the spectrum is distributed over all the real line. Let α be real, $|\alpha| > 1$, and let $\mathcal{A} \ni 0$ be a finite set of consecutive

integers. Set

$$X^{\mathcal{A}}(\alpha) = \left\{ \sum_{j=0}^n a_j \alpha^j : n \in \mathbb{N}, a_j \in \mathcal{A} \right\}.$$

With such a notation, we will prove the following statement.

Theorem 2. *Let β be a quadratic Pisot unit, and let $\mathcal{A} \ni 0$ be a finite set of consecutive integers, $\#\mathcal{A} > \beta$. Let*

$$\alpha = -\beta \quad \text{or} \quad \alpha = \beta \text{ and } \{-1, 0, 1\} \in \mathcal{A}.$$

Then there exist $\Delta_1, \Delta_2 > 0$ such that the distances between consecutive points in $X^{\mathcal{A}}(\alpha)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$. Moreover, the bidirectional sequence of gaps in $X^{\mathcal{A}}(\alpha)$ is a coding of exchange of three intervals.

The correspondence of spectra with 3iet words is established using the so-called cut-and-project scheme, introduced in the following section.

2. Cut-and-project sets

When β is an algebraic integer of degree d , the elements of the spectra are clearly expressible as elements of the set $\mathbb{Z}[\beta] = \mathbb{Z} + \mathbb{Z}\beta + \dots + \mathbb{Z}\beta^{d-1}$. If, moreover, β is a Pisot number (algebraic integer > 1 with conjugates of modulus strictly less than 1), one can easily show that the Galois image of the spectrum of a Pisot number is bounded.

In the following proposition, we determine the interval in which the Galois image of the spectrum is contained in case that β is a quadratic Pisot unit.

Proposition 3. *Let β be a quadratic Pisot unit, $m \in \mathbb{N}$. Then*

$$X^m(\beta) \subset \{x \in \mathbb{Z}[\beta] : x' \in \Omega\},$$

where x' stands for the Galois image of x in the field $\mathbb{Q}(\beta)$ and

$$\Omega = \left(-\frac{m\beta}{\beta^2 - 1}, \frac{m\beta^2}{\beta^2 - 1} \right) \text{ when } \beta^2 = p\beta + 1, \text{ and}$$

$$\Omega = \left[0, \frac{m\beta}{\beta - 1} \right) \text{ when } \beta^2 = p\beta - 1.$$

Proof. Take $x = \sum_{i=0}^n a_i \beta^i \in X^m(\beta)$. Obviously, $x \in \mathbb{Z}[\beta] = \mathbb{Z} + \mathbb{Z}\beta$. Taking the Galois image, we have $x' = \sum_{i=0}^n a_i \beta'^i$. If β satisfies $\beta^2 = p\beta + 1$, we have $\beta' = -\frac{1}{\beta}$, and thus

$$-\frac{m\beta}{\beta^2 - 1} = \sum_{i=0}^{\infty} \frac{m}{-\beta^{2i+1}} < x' = \sum_{i=0}^n \frac{a_i}{(-\beta)^i} < \sum_{i=0}^{\infty} \frac{m}{\beta^{2i}} = \frac{m\beta^2}{\beta^2 - 1}.$$

The case $\beta^2 = p\beta - 1$ is proven similarly, taking into account that $\beta' = \frac{1}{\beta}$. \square

The spectrum $X^m(\beta)$ lies in $\mathbb{Z}[\beta]$, and the pairs (x, x') for $x \in \mathbb{Z}[\beta]$ form a lattice. The above proposition states that the pairs (x, x') with $x \in X^m(\beta)$ belong to a strip of a bounded width cut from the lattice. Such considerations give a motivation for the definition of a cut-and-project set.

Definition 4. Let ε, η be irrational, $\varepsilon \neq \eta$ and Ω a bounded interval. Let $\star : \mathbb{Z}[\eta] \rightarrow \mathbb{Z}[\varepsilon]$ be the isomorphism between additive groups $\mathbb{Z}[\eta] = \mathbb{Z} + \mathbb{Z}\eta$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z} + \mathbb{Z}\varepsilon$ given by $(a + b\eta)^\star = a + b\varepsilon$. The set

$$\Sigma_{\varepsilon, \eta}(\Omega) = \{x \in \mathbb{Z}[\eta] : x^\star \in \Omega\}$$

is called a cut-and-project set with acceptance interval Ω .

Note that the above definition is a very special case of a rather general concept [19] of model sets arising by projection of a d -dimensional lattice to a suitably oriented lower-dimensional subspace of \mathbb{R}^d . As shown in [13], the sequence of gaps in a cut-and-project set can be generated using the transformation of exchange of 2 or 3 intervals.

Definition 5. Let $0 < \lambda \leq \mu < 1$. The transformation $T : [0, 1) \rightarrow [0, 1)$ defined by

$$T(x) = \begin{cases} x + 1 - \lambda & \text{for } x \in [0, \lambda) =: I_A \\ x + 1 - \lambda - \mu & \text{for } x \in [\lambda, \mu) =: I_B \\ x - \mu & \text{for } x \in [\mu, 1) =: I_C \end{cases}$$

is called an exchange of 3 intervals, if $\lambda < \mu$, (and exchange of 2 intervals, when $\lambda = \mu$).

For an arbitrary $\rho \in [0, 1)$, the orbit of ρ under T can be coded by a bidirectional infinite word $\mathbf{u} = \cdots u_{-2}u_{-1}u_0u_1u_2\cdots$ over a ternary alphabet, say $\{A, B, C\}$, (or a binary alphabet $\{A, C\}$), in a natural way,

$$u_n = X \quad \text{if } T^n(\rho) \in I_X.$$

The infinite word \mathbf{u} is called a 3iet word. If the parameters $1 - \lambda$ and μ are linearly independent over \mathbb{Q} , then the frequency of letters A, B, C in the infinite word \mathbf{u} is equal to the length of the corresponding intervals I_A, I_B, I_C , respectively. The following is a result of [13] describing the distances between consecutive elements of a cut-and-project set and their ordering.

Proposition 6. Let ε, η be irrational, $\varepsilon \neq \eta$ and Ω a bounded interval. Then there exist positive $\Delta_1, \Delta_2 \in \mathbb{Z}[\eta]$, satisfying $\Delta_2^\star < 0 < \Delta_1^\star$, such that the distances between consecutive elements of $\Sigma_{\varepsilon, \eta}(\Omega)$ take values in $\{\Delta_1, \Delta_2, \Delta_1 + \Delta_2\}$. Moreover, if the gaps Δ_1 , Δ_2 , and $\Delta_1 + \Delta_2$ are coded by letters A, C , and B respectively, then we obtain a 3iet word coding an exchange T of three intervals with parameters $\lambda = 1 - \Delta_1^\star/|\Omega|$, $\mu = -\Delta_2^\star/|\Omega|$, where $|\Omega|$ stands for the length of the interval Ω .

The distances between consecutive points of $\Sigma_{\varepsilon,\eta}(\Omega)$ take always two or three values which are given in terms of the continued fractions of ε, η , dependently on the width of the interval Ω , but not on its position on the real line. The distances are two for a discrete set of values of the width of Ω , and three otherwise.

The spectra studied in this paper will be put into connection with cut-and-project sequences where the parameters ε, η are mutually conjugated quadratic units $\eta = \beta$, $\varepsilon = \beta'$, and the isomorphism \star of the additive groups $\mathbb{Z}[\varepsilon]$, $\mathbb{Z}[\eta]$ is the Galois automorphism on the field $\mathbb{Q}(\beta)$. For simplicity, we denote the corresponding cut-and-project set $\Sigma_{\beta',\beta}(\Omega) = \Sigma_{\beta}(\Omega)$. In such a case the dependence of distances on the width of the interval Ω is expressed in a more explicit form, and we will use it in Section 7 to provide the values of distances and the corresponding frequencies in the spectra.

3. Spectrum is not equal to a cut-and-project set

Cut-and-project sets were implicitly used in [14] for the description of spectrum $X^m(\tau)$ where $\tau = \frac{1}{2}(1 + \sqrt{5})$. The proof of the result stands on a formula (given in [14] just before Definition 4), saying that a sufficiently large real number y belongs to the spectrum $X^m(\tau)$ if and only if y belongs to $\Sigma_{\tau}(\Omega)$ with Ω given in Proposition 3. Such a statement is, however, true only if $m = 1$. The aim of this section is to prove the following proposition.

Proposition 7. *Let $\beta > 1$ be a quadratic unit, $m \in \mathbb{N}$, $m \geq \lfloor \beta \rfloor$, and let Ω be as in Proposition 3. Then there exist infinitely many $y > 0$ such that $y \in \Sigma_{\beta}(\Omega)$ and $y \notin X^m(\beta)$, unless $m = \lfloor \beta \rfloor$ and $\beta^2 = p\beta + 1$ for $p \geq 1$.*

The statement of the proposition follows from Corollaries 10 and 13, which we provide below. Let us first present a suitable notation for elements of the spectrum $X^m(\beta)$, i.e. numbers which written in base β use only non-negative powers. For simplifying the notation of $y = \sum_{k=0}^{n-1} y_k \beta^k \in X^m(\beta)$ we write symbolically

$$y = y_{n-1}y_{n-2} \cdots y_1y_0 \bullet$$

and speak about a representation of y . In all this section, the base β and the parameter m are fixed, therefore we need not include them in the notation of the representation. Let us mention that a more detailed explanations to number representations will be given (and needed) in the following section.

A number $y \in X^m(\beta)$ can have many representations. For a quadratic unit base $\beta > 1$, we will use representations of a special form, as given in Lemmas 8 and 11.

The string of digits $y_{n-1}y_{n-2} \cdots y_1y_0$ can be regarded as a finite word of length n over the alphabet \mathcal{A} . The set of finite words over \mathcal{A} is denoted by \mathcal{A}^* . Together with the operation of concatenation (where the neutral element is the empty word ϵ), it is a monoid. We denote by w^i the concatenation of i copies of a finite word w , and by w^ω an infinite repetition of w . The monoid \mathcal{A}^* is linearly ordered by the usual lexicographical order \prec_{lex} . It is natural to extend

the lexicographic order to infinite words over \mathcal{A} which form a set denoted by $\mathcal{A}^{\mathbb{N}}$.

We distinguish two cases of bases. First we consider $\beta^2 = p\beta - 1$, $p \geq 3$.

Lemma 8. *Let $\beta > 1$ be a root of $x^2 - px + 1$, $p \geq 3$, and let $m \in \mathbb{N}$, $m \geq \lfloor \beta \rfloor = p - 1$. Then every $y \in X^m(\beta)$ has in base β a representation $y = ucm^j\bullet$, where*

1. $c \in \{0, 1, \dots, m - 1\}$,
2. $u \in \{0, 1, \dots, p - 1\}^*$,
3. any suffix of u is lexicographically smaller than the infinite eventually periodic word $(p - 1)(p - 2)^\omega$.

Proof. As $\beta^2 + 1 = p\beta$, we have

$$(i) \quad z = 0p0\bullet \Rightarrow z = 101\bullet,$$

$$(ii) \quad z = 0(p - 1)(p - 2)^k(p - 1)0\bullet \Rightarrow z = 10^{k+2}1\bullet \text{ for all } k \in \mathbb{N}.$$

Note that the representation of z on the right in the implications has strictly smaller sum of digits than the representation of z on the left. Let us demonstrate that by using the rules (i) and (ii) one can reduce the sum of digits in a general representation of a number $y = \sum_{j=0}^n y_j \beta^j \in X^m(\beta)$, where $y_j \in \{0, 1, \dots, m\}$.

If there is an $i \in \mathbb{N}$ such that $y_i, y_{i+2} < m$ and $y_{i+1} \geq p$, then by (i), the number $y_{i+2}y_{i+1}y_i\bullet$ has in the alphabet $\{0, 1, \dots, m\}$ also the representation $(y_{i+2} + 1)(y_{i+1} - p)(y_i + 1)\bullet$, and thus also y has another representation over $\{0, 1, \dots, m\}$ with strictly smaller digit sum. Similarly, if there exist $i, k \in \mathbb{N}$ such that $y_i, y_{i+k+3} < m$, $y_{i+1}, y_{i+k+2} \geq p - 1$ and $y_{i+j} \geq p - 2$ for $j = 2, 3, \dots, k + 1$, then by (ii),

$$(y_{i+k+3} + 1)(y_{i+k+2} - p + 1)(y_{i+k+1} - p + 2) \cdots (y_{i+2} - p + 2)(y_{i+1} - p + 1)(y_i + 1)\bullet$$

is a representation of $y_{i+k+3} \cdots y_i\bullet$ with strictly smaller digit sum, and therefore y has another representation with strictly smaller digit sum.

Repeated application of rules (i) and (ii) yields a representation of $y \in X^m(\beta)$ in the form $y = \sum_{j=0}^t r_j \beta^j$, where no other application of rules (i) or (ii) is possible. We will show that the latter representation satisfies the properties given in the lemma.

First let us show that if $m > p - 1$, then the digits m can appear only in the suffix m^j of $r_t \cdots r_0\bullet$. Indeed, if there is a pair of consecutive digits my , where $y < m$, then it occurs in a string $xm^k y$, where also $x < m$. But then, one can use the rule (i) if $k = 1$ or the rule (ii) if $k \geq 2$.

For $m \geq p - 1$, denote by j the minimal index such that $r_j < m$. If $j > t$, then we set $c = 0 = u$ and the proof is finished. If $j \leq t$ we set $c = r_j$ and $u = r_t \cdots r_{j+1}$. In order to complete the proof, we need to verify that every suffix of u is lexicographically smaller than $(p - 1)(p - 2)^\omega$. This is indeed true, since otherwise one can apply the rule (ii). \square

Remark 9. Note that numbers y with representation $y = u\bullet$, where u satisfies the properties 2. and 3. in Lemma 8, are precisely all the non-negative β -integers in the Rényi numeration system; their set is denoted by \mathbb{Z}_β^+ , see [5]. It is shown there that

$$X^{\lfloor \beta \rfloor}(\beta) \supset \mathbb{Z}_\beta^+ = \Sigma_\beta(\Omega) \cap [0, +\infty), \quad \text{where } \Omega = [0, \beta).$$

This implies that for $y = u\bullet$, one has $y' < \beta$.

The following corollary shows that the inclusion in Proposition 3 cannot be replaced by equality if only finitely many points are inserted to the spectrum.

Corollary 10. Let $\beta > 1$ be a root of $x^2 - px + 1$, $p \geq 3$, and let $m \in \mathbb{N}$, $m \geq \lfloor \beta \rfloor$. For $k \geq 1$, define $y_k = \bar{1}m^k\bullet$, where $\bar{1}$ stands for -1 . Then $y_k \in \Sigma_\beta(\Omega)$, where $\Omega = [0, \frac{m\beta}{\beta-1})$, but $y_k \notin X^m(\beta)$.

Proof. We have $y_k = \bar{1}m^k\bullet \in \Sigma_\beta(\Omega)$ since

$$0 < m - \frac{1}{\beta} = y'_1 \leq y'_k < \sum_{j=0}^{\infty} m(\beta')^j = \sum_{j=0}^{\infty} \frac{m}{\beta^j} = \frac{m\beta}{\beta-1},$$

where we have used the fact that the sequence $(y'_n)_{n \geq 1}$ is increasing.

We now show that the assumption $y_k = \bar{1}m^k\bullet \in X^m(\beta)$ leads to contradiction. For $k = 1$, we have $y_1 = m - \beta$ and

$$y'_1 = -\beta' + m = -\frac{1}{\beta} + m > m - 1. \quad (2)$$

If $y_1 \in X^m(\beta)$, then by Lemma 8 one can represent y_1 by $y_1 = ucm^j\bullet$, with the required properties. Since the digits of the new representation of ucm^j are non-negative, its last digit must be less or equal to $\lfloor y_1 \rfloor = m - p$. Thus necessarily $j = 0$ and $c \leq m - p$, i.e. $y_1 = uc\bullet$. Since by Remark 9, we have $(u\bullet)' < \beta$, we can write

$$y'_1 = (\beta(u\bullet) + c)' = \frac{(u\bullet)'}{\beta} + c < \frac{\beta}{\beta} + m - p \leq m - 2,$$

This contradicts (2).

Suppose that for some $k \geq 2$, we have $y_k \in X^m(\beta)$. Let k be minimal with this property. We claim that the last digit of the representation of y_k from Lemma 8 is equal to m . Otherwise, the representation is of the form $y_k = uc\bullet$, where $(u\bullet)' < \beta$ and $c \leq m - 1$, i.e.

$$y'_k = (\beta(u\bullet) + c)' = \frac{(u\bullet)'}{\beta} + c \leq \frac{\beta}{\beta} + m - 1 \leq m. \quad (3)$$

On the other hand, we have $y_k = \bar{1}m^k$

$$y'_k \geq y'_2 = (\bar{1}mm\bullet)' = -(\beta^2)' + m\beta' + m = \frac{-1 + m\beta}{\beta^2} + m > m,$$

which contradicts (3). Thus indeed, the representation of y_k from Lemma 8 ends in m , i.e. $y_k = ucm^j\bullet$ with $j \geq 1$. Then we can consider $y_{k-1} = \frac{y_k - m}{\beta}$ with a representation in the form $ucm^{j-1}\bullet$, which proves that $y_{k-1} \in X^m(\beta)$. This contradicts the choice of k as the minimal index such that $y_k \in X^m(\beta)$. \square

Now, we will consider the second class of quadratic unit bases, namely such that $\beta^2 = p\beta + 1$, $p \geq 1$.

Lemma 11. *Let $\beta > 1$ be a root of $x^2 - px - 1$, $p \geq 1$, and let $m \in \mathbb{N}$, $m \geq \lfloor \beta \rfloor = p$. Then every $y \in X^m(\beta)$ has in base β a representation $y = uvv\bullet$, where*

1. v is a (possibly empty) prefix of the infinite purely periodic word $(m0)^\omega$,
2. $w = \epsilon$ or $w = cd$ where $c, d \in \{0, 1, \dots, m-1\}$, and if $d \geq 1$, then $c \leq p-1$.
3. $u \in \{0, 1, \dots, p\}^*$.

Proof. If $m = \lfloor \beta \rfloor = p$, then obviously, every element of the spectrum is in the form $y = u\bullet$. Assume therefore that $m > \lfloor \beta \rfloor$. The demonstration will use methods analogous to those of the proof of Lemma 8. From $\beta^2 = p\beta + 1$, we can derive the rewriting rules

- (i) $z = 0p1\bullet \Rightarrow z = 100\bullet$,
- (ii) $z = 0(p+1)00\bullet \Rightarrow z = 10(p-1)1\bullet$.

Note that the representation of z on the right in the implication in (i) has strictly smaller sum of digits than the representation of z on the left. In the rule (ii), both representations have the same sum of digits, but the representation on the right is strictly lexicographically greater than that on the left.

Consider $y \in X^m(\beta)$. Repeated application of rules (i) and (ii) yields a representation of y in the form $y = \sum_{j=0}^t r_j \beta^j$, where no other application of rules (i) or (ii) is possible. We will show that the latter representation satisfies the properties given in the lemma.

The final representation does not contain a substring of digits cd with $c \geq p$, $d \geq 1$. Otherwise, consider the most left occurrence of such cd in the string $r_t r_{t-1} \dots r_0$, i.e. one has a factor xcd , where $x < m$. Then one can use the rule (i).

The representation $r_t r_{t-1} \dots r_0 \bullet$ does not contain a substring $m0d$ with $d < m$. By what has just been said, such a substring occurs as a suffix of $xm0d$, where $x < m$. Then one can use the rewriting rule (ii). (Note that at this point, we use the inequality $m \geq p+1 = \lfloor \beta \rfloor + 1$.)

The only occurrence of the digit m in the representation $r_t r_{t-1} \dots r_0 \bullet$ is in a suffix v , which is of the form $(m0)^j$ or $(m0)^j m$, $j \geq 0$. Let i be such that $r_t \dots r_0 = r_t \dots r_i v$, and the word $r_t \dots r_i$ has only digits in $\{0, 1, \dots, m-1\}$. Moreover, if for some $k \geq i$, we have $r_k > p$, then $k = i$ or $k = i+1$. Otherwise, we can use the rewriting rule (ii). \square

Remark 12. Note that numbers y with representation $y = u\bullet$ from Lemma 11 are in fact elements of the spectrum $X^{\lfloor \beta \rfloor}(\beta)$. Since β satisfying $\beta^2 = p\beta + 1$, $p \geq 1$, is among the so-called confluent Pisot numbers, we can use the result of [11] that any such y is in fact a β -integer (forming the set \mathbb{Z}_β^+). From [5] we then have

$$X^{\lfloor \beta \rfloor}(\beta) = \mathbb{Z}_\beta^+ = \Sigma_\beta(\Omega) \cap [0, +\infty), \quad \text{where } \Omega = (-1, \beta).$$

This implies that if $y = u\bullet$, then $-1 < y' < \beta$.

Note that for the roots of $x^2 - px - 1$ we have equality between the β -integers and the spectrum $X^m(\beta)$ with $m = \lfloor \beta \rfloor$, unlike the other class of quadratic Pisot units, see Corollary 10. The following corollary therefore does not allow $m = \lfloor \beta \rfloor$.

Corollary 13. Let $\beta > 1$ be a root of $x^2 - px - 1$, $p \geq 1$, and let $m \in \mathbb{N}$, $m > \lfloor \beta \rfloor$. For $k \in \mathbb{N}$ define $y_k = \overline{1}(0m)^k\bullet$. Then $y_k \in \Sigma_\beta(\Omega)$ where $\Omega = \left(-\frac{m\beta}{\beta^2-1}, \frac{m\beta^2}{\beta^2-1}\right)$, but $y_k \notin X^m(\beta)$.

Proof. All numbers $y_k = \overline{1}(0m)^k\bullet$, $k \in \mathbb{N}$, belong to $\Sigma_\beta(\Omega)$ since

$$\begin{aligned} -1 = y'_0 < 0 < y'_1 &= -\frac{1}{\beta^2} + m \leq y'_k = -(\beta')^{2k} + \sum_{i=0}^{k-1} m(\beta')^{2i} = \\ &= -\frac{1}{(-\beta)^{2k}} + \sum_{i=0}^{k-1} \frac{m}{(-\beta)^{2i}} < \sum_{i=0}^{\infty} \frac{m}{\beta^{2i}} = \frac{m\beta^2}{\beta^2-1}. \end{aligned}$$

Let us now show that $y_k \notin X^m(\beta)$ for all $k \in \mathbb{N}$. First realize that $y_0 = -1 \notin X^m(\beta) \subset [0, +\infty)$. For contradiction, suppose $k \geq 1$ is the minimal index such that $y_k \in X^m(\beta)$. Find a representation of y_k in the form $uvw\bullet$ as given in Lemma 11. We will use that $-1 < (u\bullet)' < \beta$, as follows from Remark 12. Let us show that this representation does not have a suffix $v = (m0)^i$. Note that $y'_k > 0$ for $k \geq 1$. We may observe that numbers of the form $y = ucdm0\bullet$ have negative Galois conjugate, namely

$$\begin{aligned} y' &= (ucdm0\bullet)' = ((u\bullet)\beta^4 + c\beta^3 + d\beta^2 + m\beta)' = \\ &= \frac{(u\bullet)'}{(-\beta)^4} + \frac{c}{(-\beta)^3} + \frac{d}{(-\beta)^2} + \frac{m}{-\beta} < \frac{\beta}{\beta^4} + \frac{m-1}{\beta^2} - \frac{m}{\beta} < 0. \end{aligned}$$

For numbers of the form $ucd(m0)^i\bullet$, $i \geq 1$, we have

$$(ucd(m0)^i\bullet)' \leq (ucdm0\bullet)' < 0.$$

This proves that the representation of y_k obtained from Lemma 11 is of the form $ucd(m0)^i m\bullet$ or $ucd\bullet$. The latter can be excluded by the following argument. Using $(u\bullet)' < \beta$, we get

$$y'_k = (ucd\bullet)' \leq (\beta^2(u\bullet) + m - 1)' = \frac{(u\bullet)'}{\beta^2} + m - 1 < \frac{1}{\beta} + m - 1. \quad (4)$$

On the other hand, since $k \geq 1$, we have

$$y'_k \geq y'_1 = (\overline{1}0m\bullet)' = m - \frac{1}{\beta^2}, \quad (5)$$

which contradicts (4), since $m - 1 + \frac{1}{\beta} \leq m - \frac{1}{\beta^2}$ for every $\beta \geq \frac{1}{2}(1 + \sqrt{5})$. We have shown that the representation of y_k is of the form $ucd(m0)^i m$ for some $i \geq 0$. We next show that the before-last digit is indeed 0. Otherwise, by item 2 of Lemma 11, the representation of y_k is $ucdm\bullet$ with $d \geq 1$, $c \leq p - 1$. We estimate

$$y'_k = (ucdm\bullet)' = m - \frac{d}{\beta} + \frac{c}{\beta^2} - \frac{(u\bullet)'}{\beta^3} < m - \frac{1}{\beta} + \frac{p-1}{\beta^2} + \frac{1}{\beta^3} = m - \frac{1}{\beta^2}, \quad (6)$$

where we use that $(u\bullet)' > -1$. Again, (5) contradicts (6). This proves that the representation of y_k has suffix $0m$. It follows that the number $y_{k-1} = \frac{1}{\beta^2}(y_k - m)$ has also a representation in the base β with non-negative digits, i.e. $y_{k-1} \in X^m(\beta)$. This is a contradiction with the choice of k as the minimal index such that $y_k \in X^m(\beta)$. \square

4. Positional representations of numbers

Consider a real basis γ , $|\gamma| > 1$, and a finite set $\mathcal{A} \ni 0$ of consecutive integers. An expression of a number $w \in \mathbb{R}$ in the form

$$w = \sum_{i=0}^{\infty} \frac{D_i}{\gamma^i}, \quad D_i \in \mathcal{A},$$

is called a (γ, \mathcal{A}) -representation of w . We usually write $w = D_0 \bullet D_1 D_2 D_3 \dots$. Denote $\mathcal{I}_{\gamma, \mathcal{A}}$ the set of real numbers w having a (γ, \mathcal{A}) -representation. If the alphabet of digits is sufficiently large, then $\mathcal{I}_{\gamma, \mathcal{A}}$ is an interval. More precisely, let $\mathcal{A} = \{a, \dots, 0, 1, \dots, A\}$, $a, A \in \mathbb{Z}$. If $A - a > |\gamma| - 1$, then

$$\mathcal{I}_{\gamma, \mathcal{A}} = \begin{cases} \left[\frac{a\gamma}{\gamma-1}, \frac{A\gamma}{\gamma-1} \right] & \text{if } \gamma > 1, \\ \left[(A + a\gamma)\frac{\gamma}{\gamma^2-1}, (A\gamma + a)\frac{\gamma}{\gamma^2-1} \right] & \text{if } \gamma < -1. \end{cases} \quad (7)$$

A variant of the above statement for positive bases can be found in [20], it can, however, be simply verified by checking that

$$\mathcal{I}_{\gamma, \mathcal{A}} = \mathcal{A} + \frac{1}{\gamma} \mathcal{I}_{\gamma, \mathcal{A}}.$$

This equality can be written in a more suitable way which gives an algorithm for finding a (γ, \mathcal{A}) -representation of a given number, namely,

$$\forall w \in \mathcal{I}_{\gamma, \mathcal{A}} \exists D \in \mathcal{A} \text{ and } \exists w_{\text{new}} \in \mathcal{I}_{\gamma, \mathcal{A}} \text{ such that } w = D + \frac{w_{\text{new}}}{\gamma}. \quad (8)$$

The digit D may not be given uniquely and almost all numbers have more than one (γ, \mathcal{A}) -representations.

In the rest of the section we provide an explicit prescription for a function $D : \mathcal{I}_{\gamma, \mathcal{A}} \rightarrow \mathcal{A}$ which allows us to find (γ, \mathcal{A}) -representations with specific properties. Besides the alphabet \mathcal{A} , we will use the alphabet $\mathcal{B} = \{b, \dots, 0, 1, \dots, B\} \subset \mathbb{Z}$ such that

$$a \leq b \leq 0 \leq B \leq A \quad \text{and} \quad B - b = \lfloor |\gamma| \rfloor. \quad (9)$$

Such an alphabet \mathcal{B} has the minimal size ensuring that $\mathcal{I}_{\gamma, \mathcal{B}}$ is an interval. Obviously

$$\mathcal{B} \subset \mathcal{A} \quad \text{and} \quad \mathcal{I}_{\gamma, \mathcal{B}} \subset \mathcal{I}_{\gamma, \mathcal{A}}.$$

We will give the prescription of the function D separately for a positive base $\gamma > 1$ and a negative base $\gamma < -1$. Both prescriptions ensure that one can find an interval I of length $|\gamma|$ for which $D(w) \in \mathcal{B}$ when $w \in I$, and I is an ‘attractor’ of the transformation $T(w) := \gamma(w - D(w))$. In particular, for every w from the interior of $\mathcal{I}_{\gamma, \mathcal{A}}$ the iterations $T^k(w)$ belong to I for sufficiently large k , and thus the corresponding (γ, \mathcal{A}) -representation of w contains eventually only digits from the alphabet \mathcal{B} . This property is formulated as Lemma 14.

We will use the notation ℓ, r for the left and right end-point of the interval $\mathcal{I}_{\gamma, \mathcal{A}}$, i.e. $\mathcal{I}_{\gamma, \mathcal{A}} = [\ell, r]$ where

$$\begin{aligned} \text{for the basis } \gamma > 1 \quad \text{one has} \quad \ell &= \frac{a\gamma}{\gamma-1}, & r &= \frac{A\gamma}{\gamma-1}, \\ \text{for the basis } \gamma < -1 \quad \text{one has} \quad \ell &= (A + a\gamma) \frac{\gamma}{\gamma^2-1}, & r &= (a + A\gamma) \frac{\gamma}{\gamma^2-1}. \end{aligned}$$

For a parameter L satisfying $\ell < L + a + 1 \leq L + A < r$, we define

$$\begin{aligned} \mathcal{I}_a &= [\ell, L + a + 1), \\ \mathcal{I}_k &= [L + k, L + k + 1), \text{ for } a < k < A, \\ \mathcal{I}_A &= [L + A, r]. \end{aligned}$$

Clearly, $\mathcal{I}_{\gamma, \mathcal{A}} = \bigcup_{k \in \mathcal{A}} \mathcal{I}_k$. The digit assignment $D : \mathcal{I}_{\gamma, \mathcal{A}} \rightarrow \mathcal{A}$ is then defined by

$$D(w) = k \quad \text{if} \quad w \in \mathcal{I}_k.$$

Denote by T the transformation

$$T(w) = \gamma(w - D(w)). \quad (10)$$

The following lemma specifies the value of L , so that T is a transformation on the interval $\mathcal{I}_{\gamma, \mathcal{A}}$, and summarizes other useful properties. We do not include the proof, which is straightforward.

Lemma 14. *With the above notation, denote by I the interval $I = \gamma \cdot [L, L + 1)$ where*

$$L = \frac{b}{\gamma - 1} \quad \text{if } \gamma > 1 \quad \text{and} \quad L = \frac{b - \gamma}{\gamma - 1} \quad \text{if } \gamma < -1.$$

Then

- (i) $T : \mathcal{I}_{\gamma, \mathcal{A}} \rightarrow \mathcal{I}_{\gamma, \mathcal{A}} ;$
- (ii) $T(\mathcal{I}_k) \subset I$ for $a < k < A ;$
- (iii) $I \subset \bigcup_{k \in \mathcal{B}} \mathcal{I}_k ;$
- (iv) $T(I) \subset I ;$
- (v) for every w belonging to the interior of $\mathcal{I}_{\gamma, \mathcal{A}}$, there exists $k \in \mathbb{N}$ such that $T^k(w) \in I$.

We will be interested in (γ, \mathcal{A}) -representations which are finite, i.e. ending in the suffix 0^ω . For numbers with finite representations, Lemma 14 implies the following statements.

Corollary 15. *Let $w \in \mathcal{I}_{\gamma, \mathcal{A}}$.*

- 1. *If $T(w)$ has a finite (γ, \mathcal{A}) -representation, then w has a finite (γ, \mathcal{A}) -representation.*
- 2. *Let γ satisfy $\gamma\mathbb{Z}[\gamma] = \mathbb{Z}[\gamma]$. Then $w \in \mathbb{Z}[\gamma] \Leftrightarrow T(w) \in \mathbb{Z}[\gamma]$.*
- 3. *Let γ satisfy $\gamma\mathbb{Z}[\gamma] = \mathbb{Z}[\gamma]$. If every $w \in I \cap \mathbb{Z}[\gamma]$ has a finite (γ, \mathcal{A}) -representation, then every $w \in \mathcal{I}_{\gamma, \mathcal{A}} \cap \mathbb{Z}[\gamma]$ has a finite (γ, \mathcal{A}) -representation.*

5. Modified spectra as cut-and-project sets

From now on, we focus on spectra of quadratic units $\alpha = \pm\beta$, where $\beta > 1$. In the proofs we will work with representations of numbers in base $\gamma = \frac{1}{\pm\beta'}$. Since β is a unit, γ is also a unit and

$$\mathbb{Z}[\beta] = \mathbb{Z}[\gamma] \quad \text{and} \quad \gamma\mathbb{Z}[\gamma] = \mathbb{Z}[\gamma].$$

The following proposition identifies the modified spectra $X^{\mathcal{A}}(\alpha)$ with cut-and-project sets. The acceptance interval Ω is in most cases given by the interior of $\mathcal{I}_{\frac{1}{\alpha'}, \mathcal{A}}$. In fact, the non-zero boundary points of $\mathcal{I}_{\frac{1}{\alpha'}, \mathcal{A}}$ have only infinite representation in base $\gamma = \frac{1}{\alpha'}$. Set

$$\Omega = \mathcal{I}_{\frac{1}{\alpha'}, \mathcal{A}}^\circ \cup \{0\}. \tag{11}$$

Proposition 16. *Let $\beta > 1$ be a quadratic unit with conjugate β' . Let $\mathcal{A} \ni 0$ be an alphabet of consecutive integers satisfying $\#\mathcal{A} > \beta$. Let Ω be given by (11). If*

$$\alpha = -\beta \quad \text{or} \quad \alpha = \beta \text{ and } \{-1, 0, 1\} \in \mathcal{A},$$

then $X^{\mathcal{A}}(\alpha) = \Sigma_\beta(\Omega)$.

Proof. Consider (γ, \mathcal{A}) -representations of numbers in $\Omega = \mathcal{I}_{\gamma, \mathcal{A}}$, where $\gamma = \frac{1}{\pm\beta'}$. Since the Galois automorphism $x \mapsto x'$ of the quadratic field $\mathbb{Q}(\beta)$ is a bijection on $\mathbb{Z}[\beta] = \mathbb{Z}[\gamma]$, equality $X^{\mathcal{A}}(\pm\beta) = \Sigma_{\beta}(\Omega)$ can be equivalently rewritten as equality of the sets

$$(X^{\mathcal{A}}(\pm\beta))' = \left\{ \sum_{k=0}^n a_k (\pm\beta')^k : n \in \mathbb{N}, a_k \in \mathcal{A} \right\} = \left\{ \sum_{k=0}^n \frac{a_k}{\gamma^k} : n \in \mathbb{N}, a_k \in \mathcal{A} \right\},$$

$$(\Sigma_{\beta}(\Omega))' = \{z' \in \mathbb{Z}[\gamma] : z' \in \Omega\} = \mathbb{Z}[\gamma] \cap \mathcal{I}_{\gamma, \mathcal{A}}.$$

As $\frac{1}{\gamma} \in \mathbb{Z}[\gamma]$, we have $\sum_{k=0}^n \frac{a_k}{\gamma^k} \in \mathbb{Z}[\gamma]$, and therefore the inclusion $(X^{\mathcal{A}}(\pm\beta))' \subset (\Sigma_{\beta}(\Omega))'$ is obvious. In order to prove the opposite inclusion, we have to show that every $w \in \mathbb{Z}[\gamma] \cap \mathcal{I}_{\gamma, \mathcal{A}}^{\circ}$ has a finite (γ, \mathcal{A}) -representation. By Corollary 15, it suffices to consider w from the attractor I of the transformation T , as defined in (10), and show that there exists $n \in \mathbb{N}$ such that $T^n(w)$ has a finite (γ, \mathcal{A}) -representation. We will even find n such that $T^n(w) \in \mathcal{B}$, where \mathcal{B} is the alphabet given in (9).

We will construct a sequence $(w_k)_{k \in \mathbb{N}}$ by the prescription $w_0 = w \in \mathbb{Z}[\gamma] \cap I$ and $w_k = T(w_{k-1})$. Obviously $w_k \in \mathbb{Z}[\gamma] \cap I$ for all $k \in \mathbb{N}$. We also have a sequence $(z_k)_{k \in \mathbb{N}}$, where $z_k := w'_k$. Since $w_{k-1} \in I$, by Lemma 14, we have

$$w_k = \gamma(w_{k-1} - D(w_{k-1})), \quad \text{where } D(w_{k-1}) \in \mathcal{B} = \{b, \dots, B\}. \quad (12)$$

In what follows, we will distinguish the proof of statement 1 and statement 2 of the proposition.

Proof of statement 1. We aim to show $X^{\mathcal{A}}(\beta) \supset \Sigma_{\beta}(\Omega)$. In this case, we have $\gamma = \frac{1}{\beta'}$, and therefore

$$z_k = \gamma'(w'_{k-1} - D(w'_{k-1})) = \frac{z_{k-1} - D}{\beta}, \quad \text{where } D \in \mathcal{B} = \{b, \dots, B\}. \quad (13)$$

Let us study the relation of numbers y and y_{new} , where y_{new} is given by $y_{\text{new}} = \frac{1}{\beta}(y - D)$ with $D \in \{b, \dots, 0, \dots, B\}$. One can easily check that

- if $y \in [-\frac{B}{\beta-1}, -\frac{b}{\beta-1}]$, then $y_{\text{new}} \in [-\frac{B}{\beta-1}, -\frac{b}{\beta-1}]$;
- if $y > -\frac{b}{\beta-1}$, then $-\frac{B}{\beta-1} < y_{\text{new}} < y$;
- if $y < -\frac{B}{\beta-1}$, then $-\frac{b}{\beta-1} > y_{\text{new}} > y$.

This, together with (13) implies that there exists $n \in \mathbb{N}$ such that

$$z_n \in [-\frac{B}{\beta-1}, -\frac{b}{\beta-1}]. \quad (14)$$

Since both w_n and z_n belong to $\mathbb{Z}[\beta] = \mathbb{Z}[\gamma]$, there exist $c, d \in \mathbb{Z}$ such that $w_n = c + d\beta$ and $z_n = c + d\beta'$. We have

$$w_n = c + d\beta \in I \quad \text{and} \quad z_n = c + d\beta' \in [-\frac{B}{\beta-1}, -\frac{b}{\beta-1}]. \quad (15)$$

Now we need to distinguish between two cases, according to the minimal polynomial of β .

- Let $\beta^2 = p\beta + 1$, $p \in \mathbb{N}$, $p \geq 1$. In this case $\gamma = \frac{1}{\beta'} = -\beta < -1$ and by Lemma 14, we have $I \subset [H, H + \beta]$, where $H = \beta \frac{b-1}{\beta+1}$. From (15), we obtain for $c, d \in \mathbb{Z}$ the inequalities

$$H \leq c + d\beta \leq H + \beta, \quad (16)$$

$$-\frac{B}{\beta-1} \leq c + d\beta' \leq -\frac{b}{\beta-1}, \quad (17)$$

In order to find $c, d \in \mathbb{Z}$ satisfying (16) and (17), realize that from the definition of H and properties $b \leq -1$, $B \geq 1$, $B - b = \lfloor \beta \rfloor$, one derives

$$H < 0, \quad H + \beta > 0, \quad -1 < -\frac{B}{\beta-1} < 0, \quad 0 < -\frac{b}{\beta-1} < 1. \quad (18)$$

If $d \geq 1$, then (16) implies that $c \leq -1$. Therefore $c - \frac{1}{\beta}d \leq -1 - \frac{1}{\beta} < -1$, which contradicts (17). Similarly, if $d \leq -1$, then (16) implies that $c \geq 1$. Therefore $c - \frac{1}{\beta}d \geq 1 + \frac{1}{\beta} > 1$, which again contradicts (17). We can therefore conclude that the only pair of integers c, d satisfying both (16) and (17) is $c = d = 0$, i.e. we necessarily have $w_n = 0$, as desired.

- Let $\beta^2 = p\beta - 1$, $p \in \mathbb{N}$, $p \geq 3$. In this case $\gamma = \frac{1}{\beta'} = \beta > 1$ and by Lemma 14, we have $I \subset [H, H + \beta]$, where $H = \frac{b\beta}{\beta-1}$. From (15), we obtain for $c, d \in \mathbb{Z}$ the same inequalities as (16) and (17), which are again valid only for $c = d = 0$.

Proof of statement 2. We need to show that $X^{\mathcal{A}}(-\beta) \supset \Sigma_{\beta}(\Omega)$. We consider $\gamma = \frac{1}{-\beta'}$. Thus from (12), we have

$$z_k = \gamma'(w'_{k-1} - D(w'_{k-1})) = \frac{z_{k-1} - D}{-\beta}, \quad \text{where } D \in \mathcal{B} = \{b, \dots, B\}. \quad (19)$$

Consider the relation of numbers y and y_{new} , which satisfy $y_{\text{new}} = -\frac{y-D}{\beta}$, with $D \in \{b, \dots, 0, \dots, B\}$. Denote $M = \max\{B, -b\}$. It can be easily checked that

- if $|y| \leq \frac{M}{\beta-1}$, then $|y_{\text{new}}| \leq \frac{M}{\beta-1}$;
- if $|y| > \frac{M}{\beta-1}$, then $|y_{\text{new}}| < |y|$;
- if $y < \frac{b}{\beta+1}$, then $y_{\text{new}} > \frac{b}{\beta+1}$;
- if $y > \frac{B}{\beta+1}$, then $y_{\text{new}} < \frac{B}{\beta+1}$;

From the first two items, we can derive that eventually, $z_k \in \left[-\frac{M}{\beta-1}, \frac{M}{\beta-1}\right]$. The latter items ensure that for some $n \in \mathbb{N}$, we have

$$\begin{aligned} z_n &\in \left[\frac{b}{\beta+1}, \frac{B}{\beta+1}\right] && \text{if } M = B, \\ z_n &\in \left[\frac{b}{\beta-1}, \frac{B}{\beta+1}\right] && \text{if } M = -b. \end{aligned} \quad (20)$$

Fix such an n , and without loss of generality consider $M = B$. If $c, d \in \mathbb{Z}$ are such that $w_n = c + d\beta$, then c, d satisfy

$$H \leq c + d\beta \leq H + \beta, \quad (21)$$

$$\frac{b}{\beta + 1} \leq c + d\beta' \leq \frac{B}{\beta - 1}, \quad (22)$$

where H is chosen so that the attractor is equal to $I \subset [H, H + \beta]$. Its value again depends on the minimal polynomial of β . It is not difficult to see that the only solution of such a system of inequalities are pairs $(c, d) = (0, 0)$ and $(c, d) = (1, 0)$, i.e. $w_n = 0$ or $w_n = 1$. \square

Proof of Theorem 2. It suffices to combine Propositions 16 stating that the modified spectrum is equal to a cut-and-project set, with Proposition 6 which describes the distances between consecutive elements of a cut-and-project set. \square

6. Distances in classical spectra

Let us return to the original question of classical spectra $X^m(\beta)$ with positive base $\beta > 1$ and alphabet of digits $\mathcal{A} = \{0, 1, \dots, m\}$. By Proposition 3, for quadratic Pisot unit β , the spectrum is included in a cut-and-project set, $X^m(\beta) \subset \Sigma_\beta(\Omega)$. Similarly as in the case of modified spectra, the acceptance interval Ω is equal to $\mathcal{I}_{\gamma, \mathcal{A}}^\circ \cup \{0\}$, where $\gamma = \frac{1}{\beta'}$.

Unlike the case of modified spectra, by Proposition 7, the inclusion $X^m(\beta) \supset \Sigma_\beta(\Omega)$ is not valid for general m , even if we restrict ourselves to any interval $[K, +\infty)$. Here, we aim to show that nevertheless, one can conclude about the values of distances in the spectrum. First we describe the elements of $\Sigma_\beta(\Omega)$ not belonging to $X^m(\beta)$.

Lemma 17. *Let $\beta > 1$ be a root of $x^2 - px + 1$, $p \geq 3$, let $m \in \mathbb{N}$, $m \geq \lfloor \beta \rfloor$. Let Ω be as in Proposition 3. Then there exists a finite set \mathcal{S} such that for every $z > 0$ satisfying*

$$z \in \Sigma_\beta(\Omega) \quad \text{and} \quad z \notin X^m(\beta)$$

there exist $j \in \mathbb{N}$ and $s \in \mathcal{S}$ such that

$$z = m \sum_{i=0}^{j-1} \beta^i + s\beta^j.$$

Proof. Consider the notation of and before Lemma 14 where we have $\gamma = \frac{1}{\beta'} = \beta$ and $\mathcal{A} = \{0, 1, \dots, m\}$, i.e. $A = m$ and $a = b = 0$. This implies that $L = 0$ and the attractor of T is thus of the form $I = [0, \beta)$. By Remark 9, if a positive $z \in \mathbb{Z}[\beta]$ satisfies $z' \in [0, \beta) = I$, then $z \in X^{\lfloor \beta \rfloor}(\beta) \subseteq X^m(\beta)$.

Suppose that $z > 0$, $z \in \mathbb{Z}[\beta]$, $z' \in \Omega = \mathcal{I}_{\gamma, \mathcal{A}}^\circ \cup \{0\}$ such that $z \notin X^m(\beta)$. In this case z' does not have a finite (γ, \mathcal{A}) -representation. Using the transformation $T(w) = \gamma(w - D(w))$ we again construct sequences $(w_k)_{k \geq 0}$, $(z_k)_{k \geq 0}$ by

the recurrence $w_0 = z'$, $w_k = T(w_{k-1})$, and

$$z_0 = z > 0, \quad z_k = w'_k = \frac{z_{k-1} - D(w_{k-1})}{\beta}.$$

By the recurrence for z_k , we see that $z_k < z_{k-1}$ when z_{k-1} is positive. Since $w_k = z'_k \in \Omega$ for all $k \in \mathbb{N}$, we have $z_k \in \Sigma_\beta(\Omega)$ which is a discrete set. Therefore there exists an index $n \in \mathbb{N}$ such that $z_n < 0 < z_{n-1}$. As $z_n = \frac{1}{\beta}(z_{n-1} - D(z'_{n-1})) < 0$, we derive $0 < z_{n-1} < D(z_{n-1}) \leq A$. In other words, z_{n-1} belongs to the set

$$\mathcal{S} := \{z \in \mathbb{Z}[\beta] : z \in (0, A), z' \in \Omega\}. \quad (23)$$

We have $\mathcal{S} = (0, A) \cap \Sigma_\beta(\Omega)$, which implies that \mathcal{S} is a finite set. By item 1 of Corollary 15, we obtain $z_{k-1} \notin X^m(\beta) \Rightarrow z_k \notin X^m(\beta)$, and thus for all $k = 0, 1, \dots, n-1$ we have $z_k > 0$ and $z_k \notin X^m(\beta)$. It follows that $z'_k \notin I$ for any $k = 0, 1, \dots, n-1$. Item (ii) of Lemma 14 together with the fact that $\mathcal{I}_0 \subset I$, guarantee that $w_k = z'_k \in \mathcal{I}_A$ for every $k = 0, 1, \dots, n-2$, whence $D(w_k) = m$ for every $k = 0, 1, \dots, n-2$. Therefore

$$w_0 = \sum_{k=0}^{n-2} \frac{D(w_k)}{\beta^k} + \frac{w_{n-1}}{\beta^{n-1}}.$$

Realizing that $w'_{n-1} = z_{n-1} \in \mathcal{S}$, we obtain $z = w'_0$ in the desired form. \square

Lemma 18. *Let $\beta > 1$ be a root of $x^2 - px - 1$, $p \geq 1$, let $m \in \mathbb{N}$, $m > \lfloor \beta \rfloor$. Let Ω be as in Proposition 3. Then there exists a finite set \mathcal{S} such that for every $z > 0$ satisfying*

$$z \in \Sigma_\beta(\Omega) \quad \text{and} \quad z \notin X^m(\beta)$$

there exist $j \in \mathbb{N}$ and $s \in \mathcal{S}$ such that

$$z = m \sum_{i=0}^{j-1} \beta^{2i} + s\beta^{2j} \quad \text{or} \quad z = m \sum_{i=0}^{j-1} \beta^{2i+1} + s\beta^{2j+1}.$$

Proof. Consider $\gamma = \frac{1}{\beta'} = -\beta$, $\Omega = \mathcal{I}_{\gamma, \mathcal{A}}$. We use the notation of Lemma 14 with $\mathcal{A} = \{0, 1, \dots, m\}$, i.e. $A = m$ and $a = b = 0$. We now have $L = -\frac{\beta}{\beta+1}$ and the attractor of T is thus of the form $I = \left(-\frac{\beta}{\beta+1}, \frac{\beta^2}{\beta+1}\right]$. In particular, $I \subset (-1, \beta)$. By Remark 12, if a positive $z \in \mathbb{Z}[\beta]$ satisfies $z' \in (-1, \beta)$, then $z \in X^{\lfloor \beta \rfloor}(\beta) \subseteq X^m(\beta)$.

Again, we construct sequences $(z_k)_{k \geq 0}$ and $(w_k)_{k \geq 0}$ by the recurrence $w_0 = z'$, $w_k = T(w_{k-1})$, and

$$z_0 = z > 0, \quad z_k = w'_k = \frac{z_{k-1} - D(w_{k-1})}{\beta}.$$

and find an index $n \in \mathbb{N}$ such that $z_n < 0 < z_{n-1}$. We define a finite set \mathcal{S} by the same prescription as in (23). From $z_0 \notin X^m(\beta)$, we have $z_k \notin X^m(\beta)$ and thus $z'_k \notin (-1, \beta)$. In particular, $z'_k \notin I$ for any $k = 0, \dots, n-1$.

By item (ii) of Lemma 14, we have $z'_k \in \mathcal{I}_0 \cup \mathcal{I}_A \setminus I$ for $k = 0, 1, \dots, n-2$. We deduce that $D(z'_k) = D(w_k) \in \{0, A\}$ for $k = 0, 1, \dots, n-2$. For concluding the proof, it suffices to show that the digits 0 and A alternate, in particular, that for $k = 0, 1, \dots, n-2$ we have

$$\begin{aligned} D(w_k) = A &\Rightarrow D(w_{k+1}) \neq A, \\ D(w_k) = 0 &\Rightarrow D(w_{k+1}) \neq 0. \end{aligned}$$

The first implication follows from $T(\mathcal{I}_A) \cap \mathcal{I}_A = \emptyset$. For the second one, recall that $w_k \notin (-1, \beta)$ for $k \in \{0, \dots, n-2\}$, and thus it suffices to verify $T(\mathcal{I}_0 \setminus (-1, \beta)) \cap \mathcal{I}_0 = \emptyset$.

Altogether,

$$w_0 = \sum_{k=0}^{n-2} \frac{D(w_k)}{(-\beta)^k} + \frac{w_{n-1}}{(-\beta)^{n-1}}$$

and $z = w'_0$ has the required form. \square

Proposition 19. *Let $\beta > 1$ be a quadratic unit, $\mathcal{A} = \{0, 1, \dots, m\}$, $m \geq \lfloor \beta \rfloor$. Let Ω be as in Proposition 3. Then for every $\delta > 0$ there exists $K > 0$ such that*

$$[K, +\infty) \cap \Sigma_\beta((1-\delta)\Omega) \subset X^m(\beta) \subset \Sigma_\beta(\Omega).$$

Proof. The inclusion on the right is given by Proposition 3. In order to prove the left inclusion, it suffices to show that for every $\delta > 0$ there exist only finitely many positive $z \in \mathbb{Z}[\beta]$ such that $z \notin X^m(\beta)$ and $z' \in (1-\delta)\Omega$. The constant K is then chosen bigger than maximum of such z .

Consider the case $\beta^2 = p\beta + 1$, $p \geq 1$. In this case $\Omega = (\frac{-m\beta}{\beta^2-1}, \frac{m\beta^2}{\beta^2-1}) = (\ell, r)$. For positive $z \in \mathbb{Z}[\beta]$ not belonging to $X^m(\beta)$ with $z' \in (1-\delta)\Omega \subset \Omega$ we can use Lemma 18, to derive that $z = x_j$ or $z = \beta x_j$, where

$$x_j = m \frac{\beta^{2j} - 1}{\beta^2 - 1} + s\beta^{2j},$$

for some $j \in \mathbb{N}$, $s \in \mathcal{S}$. For the Galois image of x_j , we have

$$x'_j = \frac{m\beta^2}{\beta^2 - 1} + \frac{1}{\beta^{2j}} \left(s' - \frac{m\beta^2}{\beta^2 - 1} \right).$$

Denote $S = \max \left\{ |s' - \frac{m\beta^2}{\beta^2-1}| : s \in \mathcal{S} \right\}$. Thus

$$x'_j > r - \frac{S}{\beta^{2j}} \quad \text{and} \quad (\beta x_j)' < \ell + \frac{S}{\beta^{2j+1}}.$$

Indices $j \in \mathbb{N}$ satisfying

$$r - \frac{S}{\beta^{2j}} < x'_j < (1-\delta)r \quad \text{or} \quad (1-\delta)\ell < (\beta x_j)' < \ell + \frac{S}{\beta^{2j+1}}$$

are only finitely many, and hence also elements $z \in \Sigma_\beta((1-\delta)\Omega)$ not belonging to the spectrum $X^m(\beta)$ are finitely many.

The proof for the case $\beta^2 = p\beta - 1$, $p \geq 3$, is analogous, using Proposition 17. \square

In the following section we will show that even if the spectrum is not equal to a cut-and-project set, Proposition 19 allows us to state that the distances take only three values. We will provide these values in an explicit form together with their frequencies.

7. Values of distances and frequencies

Sections 5 and 6 put into connection the spectra (in general $X^{\mathcal{A}}(\alpha)$) with cut-and-project sets. In order to determine the exact values of distances in $X^{\mathcal{A}}(\alpha)$ and their frequencies, let us recall a result of [18], providing a formula for the distances between consecutive points of a cut-and-project sequence $\Sigma_\beta(\Omega) = \{x \in \mathbb{Z}[\beta] : x' \in \Omega\}$, where β is a quadratic unit. We have $\beta\mathbb{Z}[\beta] = \mathbb{Z}[\beta]$, and we can derive, directly from the definition, that

$$\begin{aligned} \beta\Sigma_\beta(\Omega) &= \{\beta x \in \beta\mathbb{Z}[\beta] : x' \in \Omega\} = \{y \in \mathbb{Z}[\beta] : (\frac{y}{\beta})' \in \Omega\} = \\ &= \{y \in \mathbb{Z}[\beta] : y' \in \beta'\Omega\} = \Sigma_\beta(\beta'\Omega). \end{aligned} \quad (24)$$

Therefore it suffices to determine the distances in cut-and-project sets with acceptance intervals of length $|\Omega|$ for example within $(1, \beta]$. For cut-and-project sets with other windows, the result can be simply derived by the rescaling property (24). Denote

$$\phi_j(\beta) := \begin{cases} \beta - j, & \text{for } j = 0, 1, \dots, \lfloor \beta \rfloor - 1, \\ 1, & \text{for } j = \lfloor \beta \rfloor. \end{cases}$$

Note that $\phi_{j+1}(\beta) < \phi_j(\beta)$ and $\bigcup_{j=1}^{\lfloor \beta \rfloor} (\phi_j(\beta), \phi_{j-1}(\beta)] = (1, \beta]$. With this notation we can cite the following result of [18].

Proposition 20. *Let β be a quadratic Pisot unit. The distances between consecutive points of the cut-and-project set $\Sigma_\beta(\Omega)$ take the following values:*

- *If $|\Omega| \in (\phi_j(\beta), \phi_{j-1}(\beta))$, $j = 1, \dots, \lfloor \beta \rfloor$, then the distances are 1, $j - \beta'$ and $j + 1 - \beta'$;*
- *if $|\Omega| = \phi_{j-1}(\beta)$, $j = 1, \dots, \lfloor \beta \rfloor$, and Ω is a semi-closed interval, then the distances take two values, namely 1, $j - \beta'$.*

As a consequence, we can provide the proof of Theorem 1.

Proof of Theorem 1. As a result of Proposition 19, for each $\delta > 0$ there exist $K > 0$ such that

$$[K, +\infty) \cap \Sigma_\beta((1-\delta)\Omega) \subset X^m(\beta) \subset \Sigma_\beta(\Omega).$$

Combining Proposition 6 with Proposition 20, we can see that for sufficiently small δ , the distances between consecutive points in $\Sigma_\beta((1-\delta)\Omega)$ and in $\Sigma_\beta(\Omega)$ take the same three values. Necessarily, the same three values are taken by distances between elements of the spectra. \square

Apart the values of distances, one can be interested in the frequency of occurrence of the given value in the gap sequence. Formally, if the gap sequence is coded by an infinite word $u = u_0 u_1 u_2 \dots$ over a finite alphabet \mathcal{B} formed by symbols each corresponding to a different value of the distance, then the frequency of the symbol $X \in \mathcal{B}$ in u is given as the limit

$$\rho_X = \lim_{n \rightarrow \infty} \frac{\#\{0 \leq i < n : u_i = X\}}{n},$$

if it exists. It is a well known fact that frequencies of letters A, B, C in 3iet words, as defined in Section 2, are given by the lengths of intervals $I_A = [0, \alpha)$, $I_B = [\alpha, \beta)$, $I_C = [\beta, 1)$ in the corresponding transformation T , cf. Definition 5, namely

$$\rho_A = \alpha, \quad \rho_B = \beta - \alpha, \quad \rho_C = 1 - \beta.$$

The following theorem summarizes the results about exact values of distances and their frequencies in both classical and modified spectra. Note that the mentioned finitely many exceptions apply only in the case of classical spectrum.

Theorem 21. *Let β be a quadratic Pisot unit, and let $\mathcal{A} \ni 0$ be a finite set of consecutive integers, $\#\mathcal{A} > \beta$. If*

$$\beta^{l-1}(\beta - 1) \cdot \max\{\beta - j, 1\} < \#\mathcal{A} - 1 \leq \beta^{l-1}(\beta - 1)(\beta - j + 1)$$

for $l \geq 0$ and $j \in \{1, \dots, \lfloor \beta \rfloor\}$, then, up to finitely many exceptions, the distances in $X^{\mathcal{A}}(\pm\beta)$ take values

$$\frac{1}{\beta^l}, \quad \frac{1}{\beta^l}(j - \beta'), \quad \frac{1}{\beta^l}(j + 1 - \beta')$$

with frequencies

$$1 - \frac{\beta^{l-1}(\beta - 1)}{\#\mathcal{A} - 1}, \quad 1 - \frac{\beta^{l-1}(\beta - j)(\beta - 1)}{\#\mathcal{A} - 1}, \quad -1 + \frac{\beta^{l-1}(\beta - j + 1)(\beta - 1)}{\#\mathcal{A} - 1},$$

where β' denotes the Galois conjugate of β .

Proof. Let us first show the statement for the modified spectrum $X^{\mathcal{A}}(\alpha)$, $\alpha = \pm\beta$, satisfying assumptions of Theorem 2. By Proposition 16, the modified spectrum is equal to the cut-and-project set $\Sigma_\beta(\Omega)$ for some Ω . By Proposition 20, the distances in $\Sigma_\beta(\Omega)$ depend only on the length of the interval. For both $\alpha = \beta$ and $\alpha = -\beta$ we have

$$|\Omega| = |\mathcal{I}_{\frac{1}{\beta^l}}| = |\mathcal{I}_{-\frac{1}{\beta^l}}| = \frac{\beta}{\beta - 1}(A - a) = \frac{\beta(\#\mathcal{A} - 1)}{\beta - 1},$$

as can be verified for both $\beta^2 = p\beta + 1$ and $\beta^2 = p\beta - 1$ using (7). By Proposition 20 and the scaling property (24), we need to find $l \in \mathbb{N}$ for which $\beta^l < |\Omega| \leq \beta^{l+1}$, and find to which interval $(\phi_j(\beta), \phi_{j-1}(\beta)]$ the length $|\Omega|\beta^{-l}$ belongs.

The frequencies are derived easily by finding the parameters α, β of intervals in the corresponding exchange of three intervals. The identification of cut-and-project set with 3iet words is given in Proposition 6. Recall that the isomorphism \star applied on $\Delta_1, \Delta_2 \in \mathbb{Z}[\beta]$ is now the Galois automorphism in the field $\mathbb{Q}(\beta)$.

In order to conclude the demonstration of the statement for the classical spectra $X^m(\beta)$, it suffices to realize that by the proof of Theorem 1, the values of distances (up to finitely many exceptions) in $X^m(\beta)$ coincide with those in $\Sigma_\beta(\Omega)$, where again $|\Omega| = |\mathcal{I}_{\frac{1}{\beta^*}}|$. \square

8. Comments

- In Section 4, we have considered representations of numbers in general bases γ , $|\gamma| > 1$ and arbitrary alphabets of consecutive integers containing 0. Let us mention that in case of positive base and the digit set $\mathcal{A} = \{0, 1, \dots, m\}$, $m > \gamma - 1$, the transformation T given by Lemma 14 restricted to the attractor is homothetic to the transformation $t : [0, 1) \rightarrow [0, 1)$ providing the greedy expansion according to Rényi [21].

Similarly, if the base is $\gamma = -\beta < -1$, and the alphabet is $\mathcal{A} = \{0, 1, \dots, m\}$, $m > \beta - 1$, then T restricted to the attractor corresponds to the transformation $t : [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}) \rightarrow [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$, as considered by Ito and Sadahiro [15].

- Algorithms for arithmetic operations in systems with base α and digit set $\mathcal{A} \subset \mathbb{Z}$, $0 \in \mathcal{A}$, usually work with numbers having finitely many non-zero digits, formally, belonging to the set

$$\text{fin}_{\mathcal{A}}(\alpha) = \left\{ \sum_{i \in J} a_i \alpha^i : J \subset \mathbb{Z}, J \text{ finite}, a_i \in \mathcal{A} \right\} = \bigcup_{k \in \mathbb{N}} \alpha^{-k} X^{\mathcal{A}}(\alpha).$$

If α is an algebraic unit, then $\text{fin}_{\mathcal{A}}(\alpha) \subset \mathbb{Z}[\alpha]$. Essential is the knowledge whether $\text{fin}_{\mathcal{A}}(\alpha)$ is closed under addition and subtraction.

It can be derived that whenever $\alpha = \pm\beta$, $\beta > 1$, is a quadratic unit and $X^{\mathcal{A}}(\alpha) = \Sigma_\beta(\Omega)$, then $\text{fin}_{\mathcal{A}}(\alpha)$ is closed under addition. If, moreover, 0 lies in the interior of the interval Ω , then $\text{fin}_{\mathcal{A}}(\alpha) = \mathbb{Z}[\alpha]$ and thus $\text{fin}_{\mathcal{A}}(\alpha)$ is closed under both, addition and subtraction. The cases of α and \mathcal{A} where this happens can be read in Proposition 16.

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